

# Investigation on Finsler geometry as a generalization to curved spacetime of Planck-scale-deformed relativity in the de Sitter case

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Over the last few years, Planck-scale modifications to particles’ dispersion relation have been deeply studied for the possibility to formulate some phenomenology of Planckian effects in astrophysical and cosmological frameworks. There are some indications [arXiv:gr-qc/0611024] that Finsler geometry can provide some generalization of Riemannian geometry which may allow to account for non-trivial (Planckian) structure of relativistic particles’ configuration space. We investigate the possibility to formalize Planck-scale deformations to relativistic models in curved spacetime, within the framework of Finsler geometry. We take into account the general strategy of analysis of dispersion relations modifications in curved spacetimes proposed in [arXiv:1507.02056], generalizing to the de Sitter case the results obtained in [arXiv:1407.8143], for deformed relativistic particle kinematics in flat spacetime using Finsler formalism.

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## I. INTRODUCTION

One of the most important definitions in Euclidean geometry concerns the norm of a vector and the distance between two points: they are defined using Pythagoras' theorem. In the XIX century, Riemann generalized Gauss' ideas, introducing the concept of manifold and constructed the Riemannian geometry, in which Pythagoras theorem would be only valid at a point or along a line, but would no longer be the way of measuring distances in an open set in the manifold; instead, such measure would be performed using a general metric field. Euclidean geometry was the background of Newtonian mechanics and was invariant by Galilean boosts and rotations, but the advent of special relativity and a new kind of invariance group inspired the use of a geometry that unified space and time in a way that could keep invariant the interval between two events: such was Minkowski geometry, in which the interval represents a distance in such spacetime, that is measured by a formula similar to the Euclidean one, however allowing for positive, null or negative norms of vectors: this is a pseudo-Euclidean space. The next generalization, responsible for a relativistic description of gravity was performed by Einstein by writing General Relativity using a generalization of the Minkowski space by the same terms of the generalization from Euclidean to Riemannian geometry: spacetime was defined as a differential manifold endowed with a pseudo-Riemannian metric.

In the geometries just described, one uses a metric to define the norm of vectors and co-vectors and, then defines the length of curves and the distance between points. An important feature of Special Relativity that gains a geometrical interpretation is the dispersion relation of particles, *id est* the equation that relates energy, momentum and mass, it is defined as the norm of the four-momentum in Minkowski spacetime. Therefore the pseudo-Euclidean nature of the Minkowski metric is what allows us to have a dispersion relation of the form  $E^2 - p^2 = m^2$ . Such relation can be generalized to a curved spacetime, where the dispersion relation is defined as the norm with the curved metric, and the Minkowskian form is achieved in normal coordinates.

On the other hand, some approaches to quantum gravity phenomenology refers to the possibility of having a Modified Dispersion Relation (MDR) as a property of the semi-classical limit of quantum gravity theories, for example  $m^2 = E^2 - p^2 + \ell E^3$ , where  $\ell$  is a deformation parameter that depends on the model (which in some cases can be

assumed to be proportional to the Planck length [1]). Thus, in this context, an immediate question arises: may a MDR be a manifestation of a departure of the Riemannian<sup>1</sup> nature of spacetime at a scale sensible for quantum gravity? If spacetime was non-Riemannian in a sense that could afford for a non-quadratic norm of vectors, then we may have a MDR, but the inverse is not necessarily true, therefore we can only speculate about it.

A known branch of differential geometry that fits perfectly into this property is the so called *Finsler geometry* [2]. Basically, it is a rigorous formalization for possible non-quadratic norms of vectors, which is achieved through a four-velocity-dependent metric, but still preserving the parametrization invariance of the arc-length of curves in the manifold through the requirement of having a metric  $g(x, \dot{x})$ , homogeneous in the four-velocities. The relations between Finsler geometry and some approaches to the Quantum Gravity problem have been widely investigated [3], with special interest in models introducing some modification to particles' dispersion relation (see for instance Ref.[4] and references therein). In particular, in [5] Girelli, Liberati and Sindoni pointed out that phenomenological models implementing some kind of departure from Lorentz symmetries can be systematically formalized within the framework of Finsler geometry, hypothesizing that such a formalism may be also the correct mathematical framework to describe the so called “rainbow metrics” approach [6].

It is still unclear whether the introduction of momentum-dependent “Rainbow” metrics imposes some sort of breakdown of Lorentz symmetry or otherwise they may be suitable for a scenario with a deformation of (local) spacetime symmetries through a deformed Poincaré group as in the Deformed Special Relativity approach [7]. This issue was preliminarily treated by the authors of Ref.[5] for what concerns the fate of spacetime symmetries in MDR-inspired Finsler geometries and was explicitly considered in Ref.[8], in which it was shown that it is possible to obtain a description of modified relativistic particle kinematics satisfying both Finsler geometry and Deformed Special Relativity prescriptions within the so called  $\kappa$ -Poincaré framework [9, 10], at least at first order in the deformation parameter  $\ell \sim 1/M_P$ , where (in units such as  $c = \hbar = 1$ ) we can expect  $M_P$  to be of the order of Planck scale  $\sim 1.2 \cdot 10^{28} \text{ eV}$ . However satisfactory for what concerns the description of particles kinematics in the flat-spacetime limit, Ref.[8] leaves open the question whether this approach can be generalized to describe Deformed Relativity particles phenomenology [11–14] within a curved background. We would like here to implement this aspect, using as guidance some previous approaches [15] to  $\ell$ -deformed particles kinematics in a totally-symmetric curved spacetime, where we will denote the parameter of spacetime curvature as  $H$ .

A clear aspect presented in [5] is the lack of a Finsler metric that would be probed through spacetime inferences with massless particles, because the massless limit of the presented metric is ill-behaved. For example, this behavior prevents us from having a metric probed by neutrinos, since, even being massive, for most practical reasons they are considered massless due to their tiny rest mass. In this paper, besides generalizing some results of Ref. [5, 8], we propose a different way of calculating a four-velocity-dependent (and still partially homogeneous in the velocities) metric, that can reproduce the main features of the standard Finsler case, like the dispersion relation, the geodesics as worldlines, the presence of DSR symmetries and, furthermore, presents a well-behaved massless limit, thus allowing for a generalized Finsler-like metric for any type of particle. Such generalization will allow us to propose an expression that deepens the one of Jacob and Piran [16] for the MDR-induced time delay for particles with different energies, and to analyze the Finsler nature of spacetime using photons/neutrinos as probes. Furthermore, we will study in the last section the law of interaction between elementary particles compatible with DSR models, furnishing a description in terms of the tangent space, thus being compatible with the Finsler formalism.

### A. Notations and Hamiltonian operator

In this paper, in order to connect with the previous literature [8, 15] and express the mathematical and physical concepts in an effective and comprehensible way, we will need to rely on different notations and coordinate sets. The first one is the comoving Relative-Locality coordinate set for momentum-space  $p_\alpha$  and spacetime flat slicing coordinates  $x^\beta$ , which satisfy the relation

$$\{p_\alpha, x^\beta\} = \delta_\alpha^\beta. \quad (1)$$

In this coordinate set the Hamiltonian which formalizes the physics of particles embedded in a de Sitter-like curved spacetime, with  $H$  the parameter of curvature and  $\ell$  the deformation parameter due to Planck-scale effects, can be written as

$$\mathcal{H} = p_0^2 - p_1^2 e^{-2Hx^0} + \ell \left( \gamma p_0^3 + \beta p_0 p_1^2 e^{-2Hx^0} \right). \quad (2)$$

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<sup>1</sup> For simplicity, from now on we remove the word “pseudo” when referring to pseudo-Riemannian spaces.

We stress that for sake of simplicity we are working in  $1+1$  dimensions at first order in  $\ell$  (since the same kinematical results are preserved in a higher number dimensions). In order to describe physics in an intuitive way, we will also consider the so called *conformal time coordinatization* expressed by coordinates  $(\Omega, \Pi)$  and  $(\eta, x)$ . The relation between the previous coordinate set and this latter one is

$$\eta = \frac{1 - e^{-Hx^0}}{H}, \quad x = x^1, \quad (3)$$

$$\Omega = p_0 e^{Hx^0}, \quad \Pi = p_1. \quad (4)$$

One can verify the effectiveness of expressing physics in conformal-time coordinates by noticing that the spacetime line-element in this coordinatization can be expressed simply as  $ds_\eta^2 = (1 - H\eta)^{-2} ds_{flat-s-t}^2$ . Previous works analysing such deformed physical frameworks (see Ref.[15]), express this same Hamiltonian (2) depending on canonical variables  $(q^\mu, P_\nu)$ , deformation parameters  $\gamma$  and  $\beta$ , deformation parameter  $\ell$  and cosmological constant  $H$ . We define the energy and momentum respectively as  $P_\mu = (\Omega, \Pi)$  and spacetime coordinates as  $q^\mu = (\eta, x)$ . Therefore, in conformal coordinates we have

$$\mathcal{H}_\Omega = (1 - H\eta)^2(\Omega^2 - \Pi^2) + \ell(1 - H\eta)^3(\gamma\Omega^3 + \beta\Omega\Pi^2). \quad (5)$$

It is also easy to verify that  $\Omega$  and  $\eta$  (as well as  $\Pi$  and  $x$ ) still are conjugate variables.

The last couple of variables which will have some importance along this article is the one composed by the so called natural momenta, *id est* those that have the same functional dependence of the charges of the translation generators of our spacetime:

$$E = p_0 - Hx^1 p_1 = \Omega(1 - H\eta) - Hx\Pi \quad (6)$$

$$p = p_1 = \Pi \quad (7)$$

Using those variables the expression of the Hamiltonian (2) becomes:

$$\begin{aligned} \mathcal{H}_E &= E^2 - p^2 + 2Hp\mathcal{N} + \ell\beta Ep(p - 2H\mathcal{N}) + \ell\gamma E(E^2 + Hp\mathcal{N}) \\ &= E^2 - p^2 + 2Hp\mathcal{N} \left(1 - \ell\left(\beta - \frac{\gamma}{2}\right)E\right) + \ell(\beta Ep^2 + \gamma E^3). \end{aligned} \quad (8)$$

The physics of particles described by the deformed Hamiltonians defined in this section was widely explored in Ref.[15], however many aspects of the formalization of those phenomena seem to lack in such approaches to the study of deformed relativistic frameworks, such as metric formalism, Killing vectors and particle interactions. We would like here to provide a first exploration on how the integration of elements borrowed by the so called deformed momentum-space framework and Finsler geometry could enrich this kind of approaches to the study of deformed relativity theories, providing solid theoretical foundations to the formalism and also new suggestions on the phenomenological side.

## II. INTRODUCTION ON DEFORMED LAGRANGIAN FORMALISM AND SYMMETRIES IN THE FLAT-SPACETIME LIMIT

Finsler geometry has been introduced in previous articles [5, 8] to formalize deformed relativistic frameworks defining a so called Finsler norm, a homogenous function  $\mathcal{F}(\dot{x})$  on tangent space. Starting from the particle action in the Hamiltonian formalism in flat spacetime, performing a Legendre transform in

$$\mathcal{S} = \int \dot{x}^\alpha p_\alpha - \lambda(\mathcal{H} - m^2) d\tau, \quad (9)$$

where  $\mathcal{H}$  is Hamiltonian (2) in the limit  $H \rightarrow 0$ , and  $\lambda$  some Lagrange multiplier. In general a semi-classical metric structure is effectively encoded in Finsler geometry. On the other hand it turns out that  $\kappa$ -Poincaré group (that we are here using as foundation of a Planck-scale deformed particle dynamics in a maximally symmetric spacetime) does describe the symmetries of particles living on a de Sitter curved momentum-space [17–23] and flat spacetime (see appendix for more details on momentum-space dynamics). The two different metric formalisms Finsler metric  $g_{\mu\nu}(\dot{x})$  and momentum-space metric

$$\zeta^{\alpha\beta}(p) = \begin{pmatrix} 1 + 2\gamma\ell p_0 & 0 \\ 0 & -(1 - 2\beta\ell p_0) \end{pmatrix} \quad (10)$$

could lead to confusion on the metric structures we refer to when we will introduce concepts such as geodesics and invariant line-element. Therefore we may need here to briefly review deformed-relativistic Lagrangian formalism and

the role that those different metrics play in it, in the massless case for sake of simplicity.

The minimization of (9) provide the relation between momentum and four-velocity (here in 1+1D):

$$\dot{x}^\alpha = \lambda(\dot{x}) \{ \mathcal{H}, x^\alpha \}, \quad (11)$$

then at first order in  $\ell$  is not hard to invert this relation, finding some  $p(\dot{x})$ , which makes possible to re-express the Casimir in terms of four-velocities:

$$\mathcal{H}(\dot{x}) = \zeta_{\alpha\beta}(\dot{x}) \dot{x}^\alpha \dot{x}^\beta, \quad (12)$$

where  $\zeta_{\mu\nu}(\dot{x})$  is the inverse of momentum-space metric expressed in terms of  $\dot{x}$ . It is also possible now to find an explicit solution for  $\lambda(\dot{x})$ , by imposing  $\partial\mathcal{L}/\partial\lambda = 0$ . In the massive case this leads to

$$\lambda(\dot{x}) = \frac{1}{2} \frac{\sqrt{\zeta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}}{m}. \quad (13)$$

In the massless case, in general, it is not possible to solve  $\partial\mathcal{L}/\partial\lambda = 0$  with respect to  $\lambda$ , however taking into account (12) and (13) we can find very naturally that in the limit  $m \rightarrow 0$ ,  $\lambda \rightarrow 1/2$ . In this limit then (11) in 1+1D become

$$\begin{cases} \dot{x}^0 = p_0 + \frac{\ell}{2} (\beta p_1^2 + 3\gamma p_0^2) \\ \dot{x}^1 = -p_1 (1 - \beta\ell p_0) \end{cases}, \quad (14)$$

and their inversion gives

$$\begin{cases} p_0 = \dot{x}^0 - \frac{\ell}{2} (\beta(\dot{x}^1)^2 + 3\gamma(\dot{x}^0)^2) \\ p_1 = -\dot{x}^1 (1 + \beta\ell\dot{x}^0) \end{cases}. \quad (15)$$

Using those last results we find that the Lagrangian of our theory can be expressed as

$$\mathcal{L} = \dot{x}^\alpha p_\alpha - \frac{1}{2} \zeta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2} g_{\sigma\rho}(\dot{x}) \dot{x}^\sigma \dot{x}^\rho, \quad (16)$$

where the Lagrangian metric  $g$  is not univocally determined, as we will better discuss later, and should be chosen according to the requests of the theory. As we can see from (16) in this kind of deformed-symmetry theories we loose the uniqueness of the metric for spacetime and momentum-space. Those different metrics play a different role in the theory and one should be always careful to use the right one in the right formula. For instance for Euler-Lagrange derived relations, such as geodesic equations or Killing equations we are referring to metric  $g$ . The light cone structure is instead characterized in terms of momentum-space metric, since the on-shell relation is  $\zeta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = m^2$ . Intriguingly, this defines the invariant (see Refs[8, 22]) element for flat spacetime as  $\Delta s^2 = \zeta_{\mu\nu} \Delta x^\mu \Delta x^\nu$ .

It may be disappointing to have lost the possibility to express our theory with a unique metric, however from (16) we can still identify a couple of interesting relations:

$$\zeta_{\mu\nu}(\dot{x}) \dot{x}^\mu \dot{x}^\nu = g^{\alpha\beta}(p) p_\alpha p_\beta, \quad (17)$$

$$\zeta^{\alpha\beta}(p) p_\alpha p_\beta = g_{\mu\nu}(\dot{x}) \dot{x}^\mu \dot{x}^\nu, \quad (18)$$

which highlight the duality between spacetime and momentum-space of this theory, widely discussed in the literature (see for instance [21]).

In order to fully understand whether deformed relativistic frameworks and Finsler geometry are physically equivalent, in [8] were introduced symmetry transformations derived within the Finsler framework with the ones generated by a deformed-Poincaré group. We want to show here how those generators can be obtained *a priori* using Finsler Killing equations with metric  $g$ . For the sake of simplicity, we will here find the boost  $\mathcal{N}$  representation in the limit  $H \rightarrow 0$  and then generalize this result to the case at all orders in  $H$ . Hamiltonian (2) at order zero in  $H$  is of course invariant under translations since

$$\{p_0, \mathcal{H}\} = 0, \quad \{p_1, \mathcal{H}\} = 0, \quad (19)$$

however if we try to find the representation of the boost imposing

$$\{\mathcal{N}, \mathcal{H}\} = 0, \quad (20)$$

we obtain two possible solutions:

$$\mathcal{N}^{(1)} = x^0 p_1 + x^1 \left( p_0 + \left( \beta + \frac{\gamma}{2} \right) \ell p_0^2 + \beta \frac{\ell}{2} p_1^2 \right), \quad (21)$$

$$\mathcal{N}^{(2)} = x^0 p_1 \left( 1 - \frac{\gamma}{2} \ell p_0 \right) + x^1 \left( p_0 + \beta \ell p_0^2 + \beta \frac{\ell}{2} p_1^2 \right), \quad (22)$$

or maybe some combination of the two. Which one should we choose? A solution to this problem can be provided by the powerful Finsler formalism already explored in the formalization of deformed relativistic frameworks in Ref.[5, 8]. In Finsler geometry it is, in fact, possible to define a deformed Killing equation, which for flat spacetime has the form

$$g_{\alpha\nu} \partial_\mu \xi^\alpha + g_{\mu\alpha} \partial_\nu \xi^\alpha + \frac{\partial g_{\mu\nu}}{\partial \dot{x}^\beta} \partial_\alpha \xi^\beta \dot{x}^\alpha = 0. \quad (23)$$

We can solve those equations for a generic four-velocity-dependent metric such as

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} 1 - \gamma \ell \dot{x}^0 & -\beta_2 \ell \dot{x}^1 \\ -\beta_2 \ell \dot{x}^1 & -1 - \beta_1 \ell \dot{x}^0 \end{pmatrix} \quad (24)$$

Obtaining (see appendix for further details) a generic representation of the boost generator of the form

$$\mathcal{N} = x^0 p_1 (1 - \ell \gamma p_0) + x^1 \left( p_0 + \left( \beta + \frac{\gamma}{2} \right) \ell p_0^2 + \frac{\ell}{2} \beta p_1^2 \right) \quad (25)$$

whose Hamiltonian can be obtained though the contraction of momenta with the metric  $\tilde{g}(p)$ :

$$\mathcal{H} = \tilde{g}^{\alpha\beta}(p) p_\alpha p_\beta = p_0^2 - p_1^2 + \ell \left( (\beta_1 + 2\beta_2) p_0 p_1^2 + \gamma p_0^3 \right). \quad (26)$$

The deformed Poincaré algebra is therefore characterized by the following Poisson brackets

$$\{p_0, p_1\} = 0, \quad \{\mathcal{N}, p_0\} = -p_1 + \ell \gamma p_0 p_1, \quad (27)$$

$$\{\mathcal{N}, p_1\} = -p_0 - \ell \left( \left( \beta + \frac{\gamma}{2} \right) p_0^2 + \frac{\beta}{2} p_1^2 \right). \quad (28)$$

In equation (26) we can recognize the limit for  $H \rightarrow 0$  of (2), under the condition  $\beta = \beta_1 + 2\beta_2$ , which introduces a freedom that we will have to constrain based on the requests of the theory. In general an approach based on curved momentum-spaces work perfectly fine with a diagonal metric, *id est*  $\beta_1 = \beta$ ,  $\beta_2 = 0$  as well as with other choices. On the other hand, in order to formalize our theory (5) as a Finsler geometry we will need a homogeneous-like metric, which will force us to the choice  $\beta_1 = \beta_2 = \beta/3$ , as we will discuss further in this article.

### A. Boost and algebra at all orders in $H$

Now that we have obtained the representation of the boost generator in the  $H \rightarrow 0$  limit, knowing the form of the classical de Sitter generator

$$\mathcal{N}_H = \left( \frac{1 - e^{-2Hx^0}}{2H} - \frac{H}{2} (x^1)^2 \right) p_1 + x^1 p_0, \quad (29)$$

it is not so difficult to find its generalization at first order in  $\ell$ , *id est*:

$$\begin{aligned} \mathcal{N} = & \left( \frac{1 - e^{-2Hx^0}}{2H} (1 - \gamma \ell (p_0 - Hx^1 p_1)) - \frac{H}{2} (x^1)^2 (1 + 2\beta \ell p_0) \right) p_1 + x^1 \left( p_0 + \left( \beta + \frac{\gamma}{2} \right) \ell p_0^2 + \frac{\ell}{2} \beta p_1^2 e^{-2Hx^0} \right) + \\ & - \frac{\ell H}{2} \beta (p_0 - Hx^1 p_1) p_1 (x^1)^2. \end{aligned} \quad (30)$$

Therefore the algebra of spacetime symmetry generators  $E, p, \mathcal{N}$  at all orders in  $H$  is characterized by the following Poisson brackets:

$$\{E, p\} = Hp, \quad \{\mathcal{N}, E\} = -p + H\mathcal{N} + \ell pE \quad (31)$$

$$\{\mathcal{N}, p\} = -E - \ell \left( \frac{\beta}{2} p^2 + \left( \beta + \frac{\gamma}{2} \right) E^2 - H(\beta - \gamma) \mathcal{N} p \right). \quad (32)$$

Of course those relations are, by definition, coherent with the Poisson brackets between generators and Hamiltonian (8):

$$\{E, \mathcal{H}_E\} = \{p, \mathcal{H}_E\} = \{\mathcal{N}, \mathcal{H}_E\} = 0.$$

It should be noticed that the algebra described in this section is compatible with the well-known q-de Sitter algebra [24–26], at first order in  $\ell$ , under the choice  $\beta = -1, \gamma = 0$ .

### III. FROM HAMILTONIAN FORMALISM TO A FINSLER GEOMETRY

We started with a non-trivial local structure of spacetime that could be modeled by a curved momentum-space, which is responsible for the definition of a modified dispersion relation and a deformed picture of interactions (that will be discussed in section V). Now, we can analyze how a spacetime can emerge from these considerations, *id est* how the local structure can interfere on the geometry of the effective manifold that these high-energetic particles probe. We will follow a procedure that resembles the one presented in Refs. [5, 8]. In these references, the authors perform a Legendre transformation that links the action of a particle from the Hamiltonian to the Lagrangian formalism. Since in Special and General Relativity one identifies the action of a massive particle with the arc-length of its trajectory, the authors generalized this assumption to the case of MDRs, defining a non-quadratic norm of the particle's four-velocity vector. Which can be modeled by the well studied Finsler geometry, that is defined by a four-velocity-dependent norm. The Finsler norm satisfies the usual properties:

$$\begin{cases} \mathcal{F}(\dot{x}) \neq 0 & \text{if } \dot{x} \neq 0 \\ \mathcal{F}(\epsilon \dot{x}) = |\epsilon| \mathcal{F}(\dot{x}) \end{cases} \quad (33)$$

The second one is rather important, since it expresses the homogeneity property of the norm, which in general implies

$$\dot{x}^\mu \frac{\partial \mathcal{F}^2}{\partial \dot{x}^\mu} = 2\mathcal{F}^2. \quad (34)$$

In [5, 8] it was formalized that a MDR theory can be well expressed in terms of Finsler geometry, identifying the Finsler norm from the action integral as

$$\mathcal{S} = m \int \mathcal{F}(\dot{x}) d\tau = m \int \sqrt{g_{\mu\nu}(\dot{x}) \dot{x}^\mu \dot{x}^\nu}, \quad (35)$$

where  $g_{\mu\nu}$  is of course the Finsler metric. As we will see, this metric depends on the four-velocity and mass of the particle, however, it will not have a smooth limit for massless particles. Therefore, we propose an intermediate procedure that allows to define an action that serves both for the massive and massless cases, thus allowing a well-defined, unique metric that may describe a generalized Finsler spacetime, that is probed by high-energetic particles.

#### A. Standard Finsler metric in de Sitter spacetime

In order to generalize the procedure just described to fully symmetric curved spacetimes, our starting point is the particle action in conformal coordinates with modified dispersion relation  $\mathcal{H}_\Omega = m^2$ ,

$$\mathcal{S}[q, p, \lambda] = \int d\tau [\dot{\eta}\Omega + \dot{x}\Pi - \lambda(\mathcal{H}_\Omega - m^2)]. \quad (36)$$

This is the simple generalization of covariant mechanics in de Sitter spacetime. The equations of motion are

$$\dot{\eta} - \lambda\ell(1 - H\eta)^3(3\gamma\Omega^2 + \beta\Pi^2) - 2\lambda(1 - H\eta)^2\Omega = 0, \quad (37)$$

$$\dot{x} - 2\lambda\ell\beta(H\eta)^3\Omega\Pi + 2\lambda(1 - H\eta)^2\Pi = 0, \quad (38)$$

$$\dot{\Omega} - 2\lambda H(1 - H\eta)(\Omega^2 - \Pi^2) - 3\lambda\ell H(1 - H\eta)^2(\gamma\Omega^3 + \beta\Omega\Pi^2) = 0, \quad (39)$$

$$\dot{\Pi} = 0. \quad (40)$$

Following the same procedures of [8], *id est* performing the Legendre transformation, substituting  $p \rightarrow \dot{q}$ , the Lagrangian is

$$\mathcal{L}(q, \dot{q}, \lambda) = \frac{\dot{\eta}^2 - \dot{x}^2}{4\lambda(1 - H\eta)^2} - \frac{\ell}{8\lambda^2} \frac{\beta\dot{\eta}\dot{x}^2 + \gamma\dot{\eta}^3}{(1 - H\eta)^3} + \lambda m^2. \quad (41)$$

Minimizing this Lagrangian with respect to  $\lambda$  we get

$$\lambda = \frac{\sqrt{\dot{\eta}^2 - \dot{x}^2}}{2m(1-H\eta)} - \ell \frac{\beta \dot{\eta} \dot{x}^2 + \gamma \dot{\eta}^3}{2(1-H\eta)(\dot{\eta}^2 - \dot{x}^2)} \quad (42)$$

and thus we can identify the Finsler norm related to Hamiltonian (5) as

$$\mathcal{F}(q, \dot{q}) = \frac{\sqrt{\dot{\eta}^2 - \dot{x}^2}}{1-H\eta} - \frac{m\ell}{2} \frac{\beta \dot{\eta} \dot{x}^2 + \gamma \dot{\eta}^3}{(1-H\eta)(\dot{\eta}^2 - \dot{x}^2)}. \quad (43)$$

At this point we can make contact with geometry, identifying the semi-classical metric structure effectively encoded in the Finsler formalism. The Finsler metric is defined through a homogenous function on tangent space

$$g_{\mu\nu} = \frac{1}{2} \frac{\partial}{\partial \dot{q}^\mu} \frac{\partial}{\partial \dot{q}^\nu} \mathcal{F}^2(q, \dot{q}), \quad (44)$$

so

$$g_{00} = \frac{1}{(1-H\eta)^2} - \frac{m\ell}{2} \frac{3\beta\dot{\eta}(\dot{x})^4 - \gamma(5\dot{\eta}^3\dot{x}^2 - 2\dot{\eta}^5 - 6\dot{\eta}\dot{x}^4)}{(1-H\eta)^2(\dot{\eta}^2 - \dot{x}^2)^{5/2}}, \quad (45)$$

$$g_{11} = -\frac{1}{(1-H\eta)^2} - \frac{m\ell}{2} \frac{\beta(2\dot{\eta}^5 + \dot{\eta}^3\dot{x}^2) + \gamma(\dot{\eta}^5 + 2\dot{\eta}^3\dot{x}^2)}{(1-H\eta)^2(\dot{\eta}^2 - \dot{x}^2)^{5/2}}, \quad (46)$$

$$g_{01} = \frac{m\ell}{2} \dot{x}^3 \frac{\beta(4\dot{\eta}^2 - \dot{x}^2) + 3\gamma\dot{\eta}^2}{(1-H\eta)^2(\dot{\eta}^2 - \dot{x}^2)^{5/2}}. \quad (47)$$

It furnishes the de Sitter metric if  $\ell = 0$  and furnishes the metric of paper [8] for  $H = 0$ ,  $\gamma = 0$  and  $\beta = -1$ .

### 1. Geodesics

The analysis done so far is valid for the case of massive particles,  $m \neq 0$ , since the limit  $m \rightarrow 0$  corresponds to a singularity, implying that observables measured with such metric involving observations with particles with tiny mass, like neutrinos, would not be well defined.

Despite this fact, it is possible to solve the geodesic equation for massless particles by making  $m = 0$  in the geodesic solution just like was done in Ref. [8]. In fact, now that we have a metric for a massive particle, we can minimize the arc-length to find its world line. This way, just as it is done in [8], even though the massless limit does not exist for the definition of the metric, it is well-behaved for the expression of the geodesics.

In fact, one could solve the geodesic equation that arises by extremizing the action, which furnishes:

$$\ddot{q}^\alpha + \gamma_{\mu\nu}^\alpha \dot{q}^\mu \dot{q}^\nu + \frac{1}{2} g^{\mu\alpha} \frac{d}{dt} \left[ \left( \frac{\partial}{\partial \dot{q}^\mu} g_{\gamma\beta} \right) \dot{q}^\gamma \dot{q}^\beta \right] = \frac{\dot{\mathcal{F}}}{\mathcal{F}} \left[ \dot{q}^\alpha + \frac{1}{2} g^{\mu\alpha} \left( \frac{\partial}{\partial \dot{q}^\mu} g_{\gamma\beta} \right) \dot{q}^\gamma \dot{q}^\beta \right], \quad (48)$$

for

$$\gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\beta} (g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}). \quad (49)$$

Using the known expressions from the Finsler geometry literature, that follow from Euler theorem for homogenous functions

$$\dot{q}^\alpha \frac{\partial g_{\mu\nu}}{\partial \dot{q}^\alpha} = \dot{q}^\mu \frac{\partial g_{\mu\nu}}{\partial \dot{q}^\alpha} = \dot{q}^\nu \frac{\partial g_{\mu\nu}}{\partial \dot{q}^\alpha} = 0, \quad (50)$$

we can write the geodesic equation as

$$\ddot{q}^\alpha + \gamma_{\mu\nu}^\alpha \dot{q}^\mu \dot{q}^\nu = \frac{d}{dt} (\ln \mathcal{F}) \dot{q}^\alpha. \quad (51)$$

As this Finsler function does not depend on the coordinate  $x$ , we conclude that  $\Pi/m = \partial F / \partial \dot{x}$  is a first integral to this problem, this is equivalent to equation (40). As this formalism is invariant by reparametrization of the solution, we can choose the gauge  $\eta(\tau) = \tau$ . Doing it and substituting in the first constrain, we can solve the equation

$$\Pi = -\frac{m\dot{x}}{(1-H\eta)\sqrt{\dot{\eta}^2 - \dot{x}^2}} - \frac{m^2\ell(\beta + \gamma)\dot{\eta}^3\dot{x}}{(1-H\eta)(\dot{\eta}^2 - \dot{x}^2)^2} = -\frac{m\dot{x}}{(1-Ht)\sqrt{1 - \dot{x}^2}} - \frac{m^2\ell(\beta + \gamma)\dot{x}}{(1-Ht)(1 - \dot{x}^2)^2}. \quad (52)$$



Integrating this equation with initial condition  $x(0) = \bar{x}$  we obtain its simple solution in which incoming photons have signature  $\Pi < 0$ :

$$x(\eta) - \bar{x} = \frac{\sqrt{m^2 + \Pi^2} - \sqrt{m^2 + \Pi^2(1 - H\eta)^2}}{H\Pi} + \ell(\beta + \gamma)\eta\Pi \left(1 - \frac{H\eta}{2}\right). \quad (53)$$

The same worldline with spatial origin in  $\bar{x}$  can as well be obtained upon integration from  $\bar{x}$  to  $x(\eta)$ , imposing the on-shell relation to the Finsler norm. For a massless, on-shell particle,  $m = 0$ ,  $|\Pi| = \Omega + \mathcal{O}(\ell)$ , the above equation is

$$x(\eta) - \bar{x} = \eta - \ell(\beta + \gamma)\eta\Omega \left(1 - \frac{H\eta}{2}\right). \quad (54)$$

This is exactly the worldline found in the literature from the Hamiltonian (5), see Ref. [15]. As this is the straightforward generalization of Refs. [5, 8], it also possesses the same kind of formal difficulties with respect to the massless metric. However, it has been argued, for example in Ref. [27], that physical observables calculated, of course, by a macroscopic observer, should be performed using the classical, Riemannian metric, and in this case, there would be no problem in having such ill-defined metric, because its effect (apart from its geodesic) would be unobservable. But we raise the possibility that the natural metric “seen” by a particle, like the one we found with a Finslerian structure, should be the one considered for the calculus of observables in the new geometrical framework, since the act of expressing a metric probed by such particle should only make sense if there is an observer who probes such geometrical structure. Which allows us to a deeper conclusion that observations performed with particles with a non-trivial dispersion relation (no matter the origin of this deformation) should indicate a possible non-Riemannian structure of the spacetime where it propagates. Thereby, by generalizing the geometrical structure with which one describes a spacetime, the physical observables should change reciprocally.

### B. Generalized Finsler-like metric

In the previous section we performed a Legendre transformation on the action of a particle with a MDR until we found a version of the action that we could identify with the arc-length of the particle’s trajectory in a certain geometry. Since this was a non-quadratic function, but still invariant under reparametrizations, we could identify a Finsler function and derive a metric from it. For the massive case, there was no problem in the definition of the metric, but for the massless one, the metric was ill-defined. Such behaviour is due to the fact that the arc-length is the Nambu-Goto-like version of the action and it is well-known that it can only properly describe and only makes sense its extremization for the massive case and since the metric that would be found should depend on the mass of the particle, it is natural that such kind of approach would be problematic for  $m \rightarrow 0$  (which was not the case in Riemannian geometry).

A possible solution for such impasse is the use of an unique action for both cases. Therefore, we propose to read a metric from the *Polyakov-like action* of a point particle [29]. Such formalism was initially explored in Ref.[30].

In fact, by performing the Legendre transformation in (36), we ended up with the Lagrangian (41). As the Lagrangian is an analytic function, the action can be uniquely expressed by a Taylor expansion in the four-velocities

$$\mathcal{S}[q, \lambda] = \int d\tau \left[ \mathcal{L}|_{\dot{q}=0} + \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} \Big|_{\dot{q}=0} \dot{q}^\mu + \frac{1}{2!} \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^\mu \partial \dot{q}^\nu} \Big|_{\dot{q}=0} \dot{q}^\mu \dot{q}^\nu + \frac{1}{3!} \frac{\partial^3 \mathcal{L}}{\partial \dot{q}^\mu \partial \dot{q}^\nu \partial \dot{q}^\gamma} \Big|_{\dot{q}=0} \dot{q}^\mu \dot{q}^\nu \dot{q}^\gamma + \dots + \lambda m^2 \right], \quad (55)$$

where the zeroth and first order terms vanish as well as those of higher than the third order. Therefore the action can be expressed as

$$\mathcal{S}[q, \lambda] = \int d\tau \left[ \frac{1}{4\lambda} g_{\mu\nu}(q, \dot{q}, \lambda) \dot{q}^\mu \dot{q}^\nu + \lambda m^2 \right], \quad (56)$$

where

$$g_{\mu\nu}(q, \dot{q}, \lambda) = g_{\mu\nu}^0(q) + g_{\mu\nu}^1(q, \dot{q}, \lambda), \quad (57)$$

and we have identified

$$\frac{1}{4\lambda} g_{\mu\nu}^0 = \frac{1}{2!} \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^\mu \partial \dot{q}^\nu} \Big|_{\dot{q}=0}, \quad \frac{1}{4\lambda} g_{\mu\nu}^1 = \frac{1}{3!} \frac{\partial^3 \mathcal{L}}{\partial \dot{q}^\mu \partial \dot{q}^\nu \partial \dot{q}^\gamma} \Big|_{\dot{q}=0} \dot{q}^\gamma. \quad (58)$$

This defines a metric that depends on the point of the manifold and the four-velocity. As we will see, the field  $\lambda$  allows to distinguish between the massive and massless cases.

### 1. Massive case

If  $m \neq 0$ ,  $\lambda$  is found to be (42). If we substitute it into the action, we find

$$\mathcal{S}[q] = m \int d\tau \sqrt{g_{\mu\nu}(q, \dot{q}) \dot{q}^\mu \dot{q}^\nu}, \quad (59)$$

where the metric is

$$g_{\mu\nu} = (1 - H\eta)^{-2} \begin{pmatrix} 1 - \ell \gamma \frac{m\dot{\eta}}{\sqrt{\dot{\eta}^2 - \dot{x}^2}} & -\frac{1}{3}\ell\beta \frac{m\dot{x}}{\sqrt{\dot{\eta}^2 - \dot{x}^2}} \\ -\frac{1}{3}\ell\beta \frac{m\dot{x}}{\sqrt{\dot{\eta}^2 - \dot{x}^2}} & -1 - \frac{1}{3}\ell\beta \frac{m\dot{\eta}}{\sqrt{\dot{\eta}^2 - \dot{x}^2}} \end{pmatrix}. \quad (60)$$

From this expression, one can already realize that such metric is well-defined in the massless limit, since the extra terms behave like  $m/\sqrt{\dot{\eta}^2 - \dot{x}^2}$ ,<sup>2</sup> whereas in the standard Finsler case we had  $m/(\dot{\eta}^2 - \dot{x}^2)^{5/2}$ , which of course explodes in the massless regime.

The extremization of this action furnishes the expected geodesics (which is necessary for consistency), given by eq. (53), which coincide with those obtained using Hamilton equations for the Hamiltonian (5) subject to the mass-shell condition  $\mathcal{H} = m^2$ , in the same parametrization.

### 2. Massless case

For  $m = 0$ , the Lagrange multiplier  $\lambda$  cannot be solved. However, we can absorb it into the definition of the parameter and define the usual *affine parameter* as  $2\lambda d\tau = ds$ . With this definition, the action assumes the familiar form

$$\mathcal{S}[q] = \frac{1}{2} \int ds g_{\mu\nu}(q, q') q'^\mu q'^\nu, \quad (61)$$

where  $q' \doteq dq/ds$ , generalizing the result obtained in (16) to curved spacetimes, for the metric (24) and  $\beta_1 = \beta_2 = \beta/3$ . Moreover, the extremization of this functional, taking into account the massless condition as a fundamental one, *id est* the on-shell condition  $\mathcal{H} = 0$  written in terms of the four-velocities  $q'$  furnishes the geodesic equation (54).

### 3. Energy-momentum-dependent metric

We have shown that a natural geometrical formalism that describes the spacetime probed by particles with a Modified Dispersion Relation consists in the use of a four-velocity-dependent metric, such that the Riemannian structure is recovered in the low-energy limit. To give another appearance to this approach we can use the definition of the conjugate momenta  $P_\mu = \partial\mathcal{L}/\partial\dot{q}^\mu$  and substitute the four-velocities that appear in the metric by the energy-momentum of the particle. In fact, we end up with

$$\tilde{g}_{\mu\nu} = f^2(\eta) \begin{pmatrix} 1 - \ell\gamma\Omega/f(\eta) & \frac{1}{3}\ell\beta\Pi/f(\eta) \\ \frac{1}{3}\ell\beta\Pi/f(\eta) & -1 - \frac{1}{3}\ell\beta\Omega/f(\eta) \end{pmatrix}, \quad (62)$$

where  $f(\eta) = (1 - H\eta)^{-1}$ . This is how our formalism takes the form of a “Rainbow metric” [6], where the metric that an observer assigns to the spacetime by using high-energetic particles depends on its energy-momentum. As can be seen, once the metric is written this way, it is evident the smooth limit between the massless and massive cases, since the dependence on the mass has disappeared, remaining just the momenta one. This was not the case for the same substitution in the standard Finsler case, as can be verified in Ref. [8] already in the flat case, which is still true by adding curvature to the spacetime.<sup>3</sup>

<sup>2</sup> It should be noticed that since the two terms go to zero with the same velocity in the classical case, then at first order, in the massless limit  $\ell m/\sqrt{\dot{\eta}^2 - \dot{x}^2} \rightarrow \ell + \mathcal{O}(\ell^2)$  and the metric  $g_{\mu\nu}$  have no singular behaviour.

<sup>3</sup> Another realization of Rainbow gravity using vector dependent metrics can be found in [31] in the context of disformal transformations.

#### IV. DEFORMED RIEMANNIAN METRIC, TIME DELAY AND REDSHIFT

The Finsler geometry is a formulation in which the arc-length functional plays a central role, and from it, one can derive a metric tensor that depends homogeneously on tangent-space elements. In this paper, we review a way of defining a four-velocity-dependent metric that shares many properties of the Finsler one, but is derived from a Polyakov-like action for a particle that obeys a Modified Dispersion Relation and generates the same basic features, like the dispersion relation from a norm and geodesics. One of our achievements is the smooth limit that the metric presents when passing from the massive to the massless case.

We would like to probe the effects of such metric structure observing the travel-time through cosmological distances of a particle sufficiently energetic. For the sake of measurements, since they are performed using a macroscopic apparatus, usually it is considered the classical Riemannian metric for the definition of observables [27], for example, the proper time of an observer on Earth, which is used to derive the expression for the time-delay of high energetic photons emitted from a GRB [16]. Since the metric is Riemannian, its components do not depend on the energy/frequency of the probed particle. The formalism of Finsler geometry that we just described allows to relax this hypothesis, as we will see in this section.

The Finsler metric is a function that depends on spacetime points and vector fields, *id est* to calculate the inner product of two vectors at a given point, it is also necessary to determine a direction or a third vector at that point. Denoting  $g^v$  the metric tensor that depends on  $v \in \Gamma(TM)$ , then the inner product of vectors  $u$  and  $w$  is  $g^v_{\alpha\beta} u^\alpha w^\beta$ , it is then, necessary to specify the three vectors  $(v, u, w)$  to measure the inner product. *However, for each fixed  $v_0$ , we have that  $g^{v_0}$  is a Riemannian metric.* Therefore, it is possible to calculate a vector-directed Riemannian scalar product at a given point for a fixed  $v_0$ , if this vector is defined at that point. See for instance Ref.[32] for an analysis of vector dependent tensor calculus.

As aforesaid, in formalisms with energy-dependent or “Rainbow” metrics, this metric is considered as the one that a particle “probes”, in this formalism, however, the observer is formalized as a low energetic worldline, therefore embedded in a momentum-independent classical metric.

The formalism we describe here allows to circumvent the impossibility of the last paragraph by showing that it is possible to have a Riemannian metric, induced by a Finsler one by fixing the vector which the metric depends on as being the integral line of the particle that probes the “Rainbow” metric, *id est*  $\tilde{g} \doteq g^{v_0}$ , where  $v_0$  is the particle’s four-velocity, is the *Riemannian metric constructed by an observer by means of measurements with a particle with four-velocity  $v_0$ .*

To visualize the reasonability of this proposal, consider the following example. Suppose that the Planck scale was not so far from our experience, such that spacetime inferences could be always performed with particles that manifest modified dispersion relations. Since we are hypothesizing that the spacetime probed by such particles is deformed with respect to the standard Riemannian one by means of a momentum-dependent metric, if an observer Alice that makes inferences using these particles intends to preserve the equivalence principle, she should assign such metric structure to the spacetime manifold. For example, the geometrical locus of the photon sphere of a Black Hole (BH) is a property of the photon’s worldlines, and if she wants to preserve the equivalence principle by representing them as geodesics, they would need to be compatible with the metric structure of the spacetime surrounding a BH, which would need to be of Rainbow nature. In other words, the map of a spacetime constructed by such observer would need to inform such energy-dependent behaviour.<sup>4</sup>

So, following this principle, another example consists in the measurement of the proper-time, elapsed on Earth from the emission of a particle from a cosmological source up to its arrival at a terrestrial detector. As the spacetime geometry inferred using a particle with the same energy is determined by the deformed metric, the proper-time is calculated as  $\int \sqrt{\tilde{g}_{\mu\nu} \dot{q}^\mu \dot{q}^\nu} d\lambda$ , which in the comoving frame is  $\int \sqrt{\tilde{g}_{00}} d\eta$ , where  $\tilde{g}$  is the Rainbow metric that depends on the energy of the detected particle.<sup>5</sup>

Therefore, for the measurement of the time delay between massless particles from GRBs [11, 14] (which is one of the standard sources of constraints for MDRs), we define it analogously to the Shapiro delay [33]. In this case, the delay was calculated as the difference in the proper time of Earth between the time taken for a radar signal to reach a target and return depending if there is a massive object that deforms the spacetime geometry probed by the photon and the observer, *id est* in the sending and receiving of the signal one compares the proper times calculated in first and zeroth order in the mass parameter, for example in a Schwarzschild spacetime. Equivalently, in this model, depending on the energy of the probed particle, the induced metric will be the ordinary de Sitter one induced by a standard dispersion relation or will manifest a first order correction if the particle is sufficiently energetic, induced by

<sup>4</sup> The study of the role of the equivalence principle in Finsler geometry in theories with MDR is still at an early stage. An interested reader may find some insight in Ref.[28]

<sup>5</sup> The effect of considering a deformed metric for this calculus can be equivalently cast by absorbing this contribution in a redefinition of the emission or detection parameter  $\eta$  and considering the standard Riemannian metric.

the corresponding MDR, which is similar to the first order correction in the mass that deforms the flat metric in the Shapiro delay. In our case, the parameter is not the mass of the Schwarzschild metric, but it is the Planck length  $\ell$ .

### A. On the propagation time

What we define as a time delay for the arrival of photons is the difference between the proper times (measured on Earth) that record the time elapsed from the emission of massless particles at  $A$ , to their arrival at the detector, labeled as  $B$ . For each photon, the proper time elapsed is

$$\tau = \int_A^B \sqrt{\tilde{g}_{00}} d\eta. \quad (63)$$

In this case,  $\tilde{g}_{00}$  is the metric's component in the deformed Riemannian geometry induced by a Finsler-like metric that the observer associates to the spacetime from eq. (62), where the photon's four-velocity/momenta is the fixed vector that induces the deformed Riemannian metric. In the coordinates that we have been using  $\tilde{g}_{00} = (1 - H\eta)^{-2} - \ell\gamma\Omega(1 - H\eta)^{-1}$ . Integrating the proper time,

$$\tau = \frac{1}{H} \ln \left| \frac{1 - H\eta_A}{1 - H\eta_B} \right| - \ell \frac{\Omega}{2} \gamma (\eta_B - \eta_A). \quad (64)$$

For sake of simplicity we assume the classical expression for the redshift  $z$

$$1 + z = a(t_B)/a(t_A) = \frac{1 - H\eta_A}{1 - H\eta_B}, \quad (65)$$

in order to keep explicit the Planck scale corrections in the time-delay formulation. An approach assuming a non-trivial expression for  $z$  can be found for instance in [26].

### B. Calculating the time delay for the perturbed metric

Now, consider that we have a hard and a soft massless particle, for which  $\Omega^s \ell \ll 1$ , and  $\Omega^h \ell \lesssim 1$ , their worldlines will be

$$x_h(\eta) = \eta - \ell(\beta + \gamma)(\eta - \eta_A)\Omega^h \left[ 1 - \frac{H(\eta - \eta_A)}{2} \right], \quad (66)$$

$$x_s(\eta) = \eta. \quad (67)$$

Suppose that both particles are emitted simultaneously, *id est*  $\eta_{sA} = \eta_{hA} \doteq -\eta_A$ , at the same spatial point  $x_s(\eta = \eta_A) = x_h(\eta = \eta_A) = x_A$  (see Fig.IV B). Let  $B$  be the point where the detector on Earth is. The elapsed parameter necessary for the soft particle to reach the detector, which is located at the origin is  $\eta_A$ .

To determine the elapsed parameter, denoted by  $\eta_{hB}$ , necessary for the hard trajectory to reach the same spacial point, we consider

$$\begin{aligned} x_h(\eta_{hB}) &= 0 \Rightarrow \\ \eta_{hB} &= \ell(\beta + \gamma)\Omega^h \left( 1 + \frac{H\eta_A}{2} \right) \eta_A. \end{aligned} \quad (68)$$

Since we normalize the scale-factor for  $\eta_B = 0$ , and the redshift  $z$  is an experimental parameter determined by soft particles, it is natural to define it as

$$1 + z = 1 - H\eta_{sA} = 1 + H\eta_A \Rightarrow \eta_A = \frac{1}{H}z. \quad (69)$$

So, the proper time elapsed on Earth until the arrival of the hard particle can be calculated using equations (68) and (69) as

$$\begin{aligned} \tau_h &= \frac{1}{H} \ln \left| \frac{1 - H\eta_{hA}}{1 - H\eta_{hB}} \right| - \ell \frac{\Omega^h}{2} \gamma (\eta_{hB} - \eta_{hA}) = \frac{1}{H} \ln \left| \frac{1 + H\eta_A}{1 - H\eta_{hB}} \right| - \ell \frac{\Omega^h}{2} \gamma (\eta_{hB} + \eta_A) \Rightarrow \\ \tau_h &\approx \tau_s + \frac{\ell\Omega^h}{H} \left[ (\beta + \gamma)(z + \frac{z^2}{2}) - \frac{\gamma}{2}z \right]. \end{aligned} \quad (70)$$

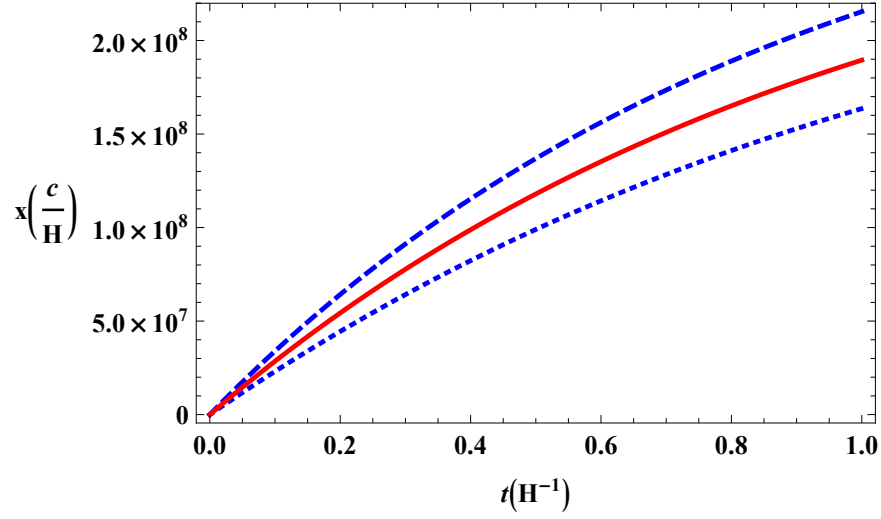


Figure 1. Two massless particles, a high energetic one (blue lines) and a low energetic one (red line), are emitted simultaneously at the origin. We can observe that the high energetic particle anticipates (dashed line) or follows (dotted line) the infrared one, whether the sum of parameters  $\beta$  and  $\gamma$  is respectively  $\beta + \gamma < 0$  or  $\beta + \gamma > 0$ .

The time delay is defined as the difference between the proper times  $\tau_h$  and  $\tau_s$ , therefore it is

$$\Delta\tau \doteq \frac{\ell\Omega^h}{H} \left[ (\beta + \gamma) \left( z + \frac{z^2}{2} \right) - \frac{\gamma}{2} z \right]. \quad (71)$$

Such expression contains the usual one used for the calculus of time-delays presented in [16], added to an extra contribution due to the deformed metric that the observer assigns to spacetime when performing measurements with high-energetic particles. The previous approach could not distinguish between the parameters  $\gamma$  and  $\beta$ , since the effect appeared as a factor of  $\gamma + \beta$ . Our new proposal allows to distinguish them, and introduces a phenomenological rule for the time-delay in a more flexible shape

$$\Delta\tau \doteq \frac{\ell\Omega^h}{H} (\xi_1 z + \xi_2 z^2). \quad (72)$$

In our case,  $\xi_1 = \beta + \gamma/2$  and  $\xi_2 = (\gamma + \beta)/2$ .

### C. Energy dependent redshift effect

Another interesting correction is the redshift of the frequency of photons in this geometry. Different from the previous case in which the metric played a central role, for calculating this quantity it is only necessary to consider the equations of motion of the photon. In fact, in comoving coordinates<sup>6</sup>, the equations of motion for spacetime coordinates and the energy for a massless particle are (14)

$$\begin{cases} \dot{x}^0 = p_0 + \frac{\ell}{2} \left( \beta p_1^2 e^{-2Hx^0} + 3\gamma p_0^2 \right) \\ \dot{x}^1 = -p_1 e^{-2Hx^0} (1 - \beta \ell p_0) \\ \dot{p}_0 = -H p_0^2 (1 + \ell \gamma p_0) \end{cases}. \quad (73)$$

The solution for the energy is

$$p_0(\tau) = \frac{\bar{p}_0}{1 + H\bar{p}_0\tau} - \ell\gamma \frac{\bar{p}_0^2}{(1 + H\bar{p}_0\tau)^2} \ln(1 + H\bar{p}_0\tau), \quad (74)$$

where  $\bar{p}_0$  is the energy of the photon for  $\tau = 0$ .

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<sup>6</sup> Now we use comoving coordinates because they are the standard ones used in the literature to measure this effect [21].

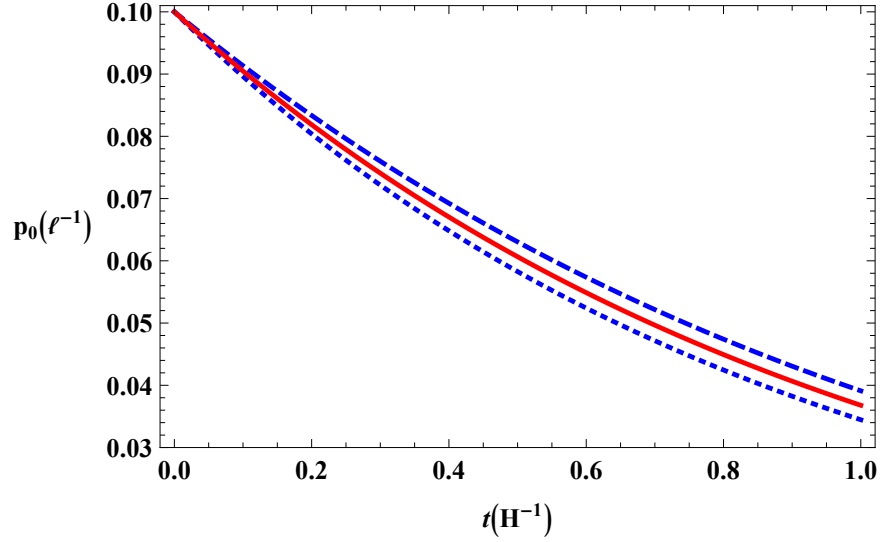


Figure 2. Particles with different energies experience a different cosmological redshift. The dashed-blue line shows the deformation effect for  $\beta + \gamma > 0$ , while the dotted-blue one for  $\beta + \gamma < 0$ . The red line represents the classical undeformed cosmological redshift effect, for a low-energetic particle.

For on-shell massless particles we have  $\mathcal{H} = 0$ , which is simply

$$p_1^2 = e^{2Hx^0} p_0^2 [1 + \ell(\beta + \gamma)p_0]. \quad (75)$$

Substituting equation (74) and (75) in the first equation of (73), we find

$$\Delta x^0(\tau) = \frac{1}{H} \ln(1 + H\bar{p}_0\tau) + \ell \frac{\bar{p}_0}{1 + H\bar{p}_0\tau} \left[ \frac{1}{2}(\beta + \gamma)\bar{p}_0\tau + \frac{\gamma}{H} \ln(1 + H\bar{p}_0\tau) \right], \quad (76)$$

which can be solved for the parameter  $\tau$  as

$$\tau = \frac{e^{H\Delta x^0} - 1}{H\bar{p}_0} - \ell \left[ (\beta + \gamma) \frac{e^{H\Delta x^0} - 1}{2H} + \gamma \Delta x^0 \right]. \quad (77)$$

Substituting this expression in the dependence of the energy with the parameter  $\tau$  given by (74), we derive the cosmological redshift of the frequency of a photon that obeys a modified dispersion relation (see Fig. IV C)

$$p_0 = \bar{p}_0 e^{-H\Delta x^0} + \frac{\ell}{2} \bar{p}_0^2 e^{-2H\Delta x^0} (\beta + \gamma) (e^{H\Delta x^0} - 1). \quad (78)$$

## V. NONTRIVIAL MOMENTUM COMPOSITION RULES

It was explained in [8], using an argument already introduced in [23], that if one assumes deformed spacetime symmetries, then if we assume that whenever two particles interact, different observers must agree on the existence of the interaction vertex, this latter may be deformed too. This feature is formalized in the  $\kappa$ -Poincaré framework, introducing a deformed composition law for momenta

$$k_\mu = (p \oplus q)_\mu, \quad (79)$$

which we can imagine as describing the decaying of a particle with momentum  $k$  in two different particles with momenta  $p$  and  $q$ . At first order in the deformation parameter  $\ell$  we can express all the possible composition rules of momenta (requiring invariance under the parity transformation  $p_0 \rightarrow p_0$ ,  $p_1 \rightarrow -p_1$ , see [34]) relying just on four parameters  $a, b, f, g$  as

$$(p \oplus q)_0 = p_0 + q_0 + \ell(ap_0q_0 + bp_1q_1), \quad (80)$$

$$(p \oplus q)_1 = p_1 + q_1 + \ell(fp_0q_1 + gp_1q_0). \quad (81)$$

Such deformed composition rules are known to be related to the coproduct of translation generators. Then, in order to determine which coalgebraic sector can be compatible with our deformed boost generator (25), we need to find some condition which would allow us to find some relations between the composition rule parameters. Those relations can be found by requiring the composition of the boosted momenta to be invariant under the action of the boost

$$O' = O + \xi\{\mathcal{N}, O\}. \quad (82)$$

However, before checking whether different (boosted) observers agree on the existence of the particles' vertex, we should pay attention to the possibility to have the so called *backreaction* effect [23, 34, 35], since the rapidity parameter  $\xi$  can in general change in a momentum-dependent way, compatible with the coproducts of momenta and the action of Lorentz transformations on momenta themselves. This effect is defined as the right action  $\triangleleft: \mathbb{R} \times \Sigma \rightarrow \mathbb{R}$ , that we can parametrize as

$$\xi_1 = \xi \triangleleft q = \xi(1 + \ell C q_0 + \ell D q_1), \quad \xi_2 = \xi \triangleleft p = \xi(1 + \ell B p_0 + \ell A p_1). \quad (83)$$

Therefore, in general we will have

$$(p \oplus q)'_{\mu}(\xi) \neq p'(\xi) \oplus q'(\xi), \quad (84)$$

$$(p \oplus q)'_{\mu}(\xi) = p'(\xi \triangleleft q) \oplus q'(\xi \triangleleft p). \quad (85)$$

Now, imposing the complete parametrization of vertex transformation (85) with (82) and (83), we are able to find four relations between our eight parameters  $a, b, f, g, A, B, C, D$ , which ensure the existence of the vertex for any observer:

$$\begin{cases} A = 0 \\ D = 0 \\ a = -\gamma \\ b = \beta - B - C \\ f = \beta - C \\ g = \beta - B \end{cases} \quad (86)$$

Those relations are perfectly compatible with the Hamiltonian invariance under boost:

$$\mathcal{H}(p \oplus q) = \mathcal{H}(k) = \mathcal{H}'(k) = \mathcal{H}(k') = \mathcal{H}(p'(\xi \triangleleft q) \oplus q'(\xi \triangleleft p)), \quad (87)$$

which shows *a posteriori* the coherence of those composition laws with the curved momentum-space framework. The problem is if also Finsler geometry can be compatible to such “deformed sum”. In Finsler geometry, in fact, the particle trajectory depends on spacetime coordinates  $x^{\alpha}$  and four-velocities  $\dot{x}^{\beta}$ , which live in a flat tangent space to the curved spacetime. On the other hand, in Relative Locality [18, 20, 21] momenta are the fundamental observables and spacetime is defined as the cotangent bundle to the curved momentum-space. If the deformation of momenta composition law fits very well with the curved momentum-space framework, one should not give for granted for this to happen also in the Finsler case. However in this section we showed that non trivial composition laws are imposed by the deformation of spacetime symmetries that arise from our theory. The impossibility to express this feature also for Finsler formalism would then jeopardise our entire argument.

### A. Four-velocity tangent space

From our perspective, the problem of the tangent space flatness is not a real problem. In fact every local observer is flat, and curvature arises only when confronting observations between different reference frames. Assuming then (79), we are treating different particles as different reference frames in momentum-space, which can be characterized also as a non-trivial composition for four-velocities, formalizing the relations between the different locally-flat reference frames. This feature can be obtained taking into account a two-parameters family of coproducts and momenta compositions that we can obtain from (86):

$$\Delta p_0 = p_0 \otimes \mathbf{1} + \mathbf{1} \otimes p_0 - \ell(\gamma p_0 \otimes p_0 - (\beta - B - C)p_1 \otimes p_1), \quad (88)$$

$$\Delta p_1 = p_1 \otimes \mathbf{1} + \mathbf{1} \otimes p_1 + \ell((\beta - C)p_0 \otimes p_1 + (\beta - B)p_1 \otimes p_0), \quad (89)$$

and

$$(p \oplus q)_0 = p_0 + q_0 - \ell(\gamma p_0 q_0 - (\beta - B - C)p_1 q_1), \quad (90)$$

$$(p \oplus q)_1 = p_1 + q_1 + \ell((\beta - C)p_0 q_1 + (\beta - B)p_1 q_0). \quad (91)$$

Then, using the expression of four-velocities with respect to momenta (14) with (90) and (91), in the massless case, we can easily obtain the composition laws for  $\dot{x}$ :

$$\dot{x}^0(p) \oplus_{\dot{x}} \dot{x}^0(q) = \dot{x}^0(p \oplus q) = \dot{x}^0(p) + \dot{x}^0(q) + 2\gamma\ell\dot{x}^0(p)\dot{x}^0(q) + \ell(2\beta - B - C)\dot{x}^1(p)\dot{x}^1(q), \quad (92)$$

$$\dot{x}^1(p) \oplus_{\dot{x}} \dot{x}^1(q) = \dot{x}^1(p \oplus q) = \dot{x}^1(p) + \dot{x}^1(q) - \ell B\dot{x}^1(p)\dot{x}^0(q) - \ell C\dot{x}^0(p)\dot{x}^1(q). \quad (93)$$

Those laws satisfy the relation

$$\dot{x}^\alpha((p \oplus q)'(\xi)) = \dot{x}^\alpha(k') = \dot{x}^\alpha(p'(\xi \triangleleft q) \oplus q'(\xi \triangleleft p)) = (\dot{x}^\alpha(p))'(\xi \triangleleft q) \oplus_{\dot{x}} (\dot{x}^\alpha(q))'(\xi \triangleleft p) = (\dot{x}^\alpha(p) \oplus_{\dot{x}} \dot{x}^\alpha(q))'(\xi), \quad (94)$$

where the backreaction has the exact same form of the one described in the previous section. A nice property that we can find *a posteriori* in the massless case is

$$\frac{\dot{x}^1(p) \oplus_{\dot{x}} \dot{x}^1(q)}{\dot{x}^0(p) \oplus_{\dot{x}} \dot{x}^0(q)} = 1 - (\beta + \gamma)\ell(p_0 + q_0) \equiv v(p \oplus q), \quad (95)$$

in which  $v$  is the velocity that we already found in our massless worldlines (54). This structure of relations between different tangent spaces allows us to identify the phase-space invariant element as:

$$\begin{aligned} \zeta_{\alpha\beta}(p \oplus q)\dot{x}^\alpha(p \oplus q)dx^\beta &= \zeta_{\alpha\beta}(p' \oplus q')\dot{x}^\alpha(p' \oplus q')d(x^\beta)' = \zeta_{\alpha\beta}((p \oplus q)')\dot{x}^\alpha((p \oplus q)')d(x^\beta)' = \\ &= \zeta_{\alpha\beta}((p \oplus q)')((\dot{x}(p))' \oplus_{\dot{x}} (\dot{x}(q))')^\alpha d(x^\beta)'. \end{aligned} \quad (96)$$

The doubt we had at the beginning of this section was whether it is possible to express departures from Poincaré symmetries, often formalized in the literature as a deformed Hamiltonian framework ( $\ell$ -deformed phase space), within the context of the Lagrange-Finsler geometry (tangent space). As commented in [5], Finsler geometry could be used to describe both breakdown and deformation of spacetime symmetries. However we have showed here that this symmetry deformation implies the necessity to characterize velocities as living in a tangent space with non-trivial translation laws. Those laws moreover result to be compatible with the phase-space deformation of the theory (2).

## VI. CONCLUSIONS

The question of what may be the nature of the spacetime emerging from a semi-classical limit of a quantum description of the gravitational degrees of freedom is still open. Possibly, we may have a dimensional reduction coming from CDT, Hořava-Lifshitz gravity and others (see [36] and references therein), non-local metrics [37], Rainbow-like metrics coming from a quantization of geometrical and matter-like degrees of freedom [27] and, as we have shown, a possible Finsler nature coming from a MDR. In all these cases there exists a common point: the non-Riemannian nature of spacetime. Which is not a complete surprise, after all there is no *a priori* request for nature of spacetime to be limited to Riemannian geometry at all scales, despite the abundance of well-defined geometrical structures (not to mention those that are still to be discovered).

In this paper, we analyzed the possibility of having a deformed dispersion relation within a curved background, investigating what kind of manifold could be the most natural to “host” them. We generalized the results presented in [5, 8] and we proposed a solution for the massless limit problem, *id est* we found a candidate partially-homogeneous metric using, which are possible to formalize, the most important properties that a proper geometrical model for the description of elementary particles should provide, such as the geodesic equation and the dispersion relation. The presence of such unified Finsler structure that an observer would assign to spacetime, when performing measurements observing high-energetic particles, allowed us to formalize the usual phenomenology of time delays of photons from GRBs due to the energy-dependent speed of light presented in [16], plus a correction of the order of the Planck length originated from the non-Riemannian structure of spacetime, with these both contributions having the same geometrical origin.

We also estimated the correction that such a model could bring to the cosmological redshift effect. In this context, it would be extremely interesting to test such effect using cosmological data, however unfortunately the best observations on some kind of energy dependence for the cosmological redshift yet published [38] do not have the sensitivity to discriminate the kind of effect we found here. The authors found no energy dependence with a precision of  $\Delta z \sim 10^{-6}$ , whereas in the visible range of electromagnetic radiation ( $\sim 5000\text{\AA}$ ) the effect we take into account would be of the order of  $\Delta z \sim 10^{-28}$  on the redshift magnitude. In any case, we need phenomenological analysis with more energetic particles.

The last open issue we explored is the deformation of velocities composition law in the Finsler tangent space that symmetrical  $\ell$ -deformed theories require (as already pointed out in [8]), in order to allow to different observers to agree on the occurrence (or not) of some interaction event. We found that introducing a slightly more complicated formalism inspired by Hopf algebras literature it is still possible to obtain a full relativistic description for particles interactions.



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### Appendix A: Connection with Relative Locality

First of all let us define a generic<sup>7</sup> momentum-space metric depending on our two parameters  $\beta, \gamma$ , at first order in the deformation parameter  $\ell$ :

$$\zeta_\ell^{\mu\nu}(P) = \begin{pmatrix} 1 + 2\gamma\ell P_0 & 0 \\ 0 & -e^{-2Hx^0}(1 - 2\beta\ell P_0) \end{pmatrix}. \quad (\text{A1})$$

As well established in literature [20–22] we can find the Hamiltonian by integrating the momentum-space invariant line-element

$$\mathcal{D}p = \int_0^1 ds \sqrt{\zeta^{\alpha\beta}(P) \dot{P}_\alpha \dot{P}_\beta}, \quad (\text{A2})$$

where the momentum-space geodesics  $P(s)$  are defined by the geodesic equation:

$$\ddot{P}_\alpha + \Gamma_\alpha^{\beta\gamma} \dot{P}_\beta \dot{P}_\gamma = 0, \quad (\text{A3})$$

in which the connections are our usual momentum-space Christoffel symbols:

$$\Gamma_\lambda^{\mu\nu} = \frac{1}{2} \left( \frac{\partial}{\partial p_\mu} \zeta^{\sigma\nu} + \frac{\partial}{\partial p_\nu} \zeta^{\sigma\mu} - \frac{\partial}{\partial p_\sigma} \zeta^{\mu\nu} \right). \quad (\text{A4})$$

Given our generic metric, then, we just have to solve the following equations:

$$\begin{aligned} \ddot{P}_0 + \ell \left( \gamma \dot{P}_0^2 - \beta e^{-2Hx^0} \dot{P}_1^2 \right) &= 0, \\ \ddot{P}_1 - 2\ell\beta \dot{P}_0 \dot{P}_1 &= 0. \end{aligned} \quad (\text{A5})$$

Since  $\ddot{P} \sim O(\ell^2)$ , we observe that in our case the geodesic can be expressed as

$$P_\alpha(s) \simeq \bar{p}_\alpha + s \dot{P}_\alpha|_{s=0} + \frac{1}{2} s^2 \ddot{P}_\alpha|_{s=0}. \quad (\text{A6})$$

Let us now impose the initial values for the geodesic as  $P_\alpha(0) = 0$  and  $P_\alpha(1) = p_\alpha$ . It is easy to notice that they imply  $\bar{p}_\alpha = 0$  and  $p_\alpha = \dot{P}_\alpha|_{s=0} + \frac{1}{2} \ddot{P}_\alpha|_{s=0}$ . We have now all we need in order to solve our equations, *id est*:

$$\begin{aligned} \dot{P}_0|_{s=0} &= p_0 + \frac{\ell}{2} \left( \gamma p_0^2 - \beta e^{-2Hx^0} p_1^2 \right), \\ \dot{P}_1|_{s=0} &= p_1 - \ell\beta p_0 p_1, \end{aligned} \quad (\text{A7})$$

and also

$$\begin{aligned} \ddot{P}_0|_{s=0} &= -\ell \left( \gamma p_0^2 - \beta e^{-2Hx^0} p_1^2 \right), \\ \ddot{P}_1|_{s=0} &= +2\ell\beta p_0 p_1. \end{aligned} \quad (\text{A8})$$

Therefore we are now able to solve integral (A2), and find the generic Hamiltonian expression:

$$\mathcal{H} = p_0^2 - p_1^2 e^{-2Hx^0} + \ell \left( \gamma p_0^3 + \beta p_0 p_1^2 e^{-2Hx^0} \right). \quad (\text{A9})$$

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<sup>7</sup> This one of course is not the most generic momentum-space metric one can possibly define, however for the purposes of this article we will not need any further parametrization.

## Appendix B: Killing equations in the flat-spacetime limit

In the introductory section we have shown that the Lagrangian-Hamiltonian Legendre transformation defines the relation

$$\zeta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 2p_\alpha\dot{x}^\alpha - \tilde{g}_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma. \quad (\text{B1})$$

Since eq. (12) basically states that the first term of B1 must be invariant under boost, then we have that

$$\delta(2p_\alpha\dot{x}^\alpha - \tilde{g}_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma) = \{\mathcal{N}, \mathcal{H}(\dot{x})\} = 0. \quad (\text{B2})$$

Therefore

$$\delta(\tilde{g}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu) = 2p_\alpha\delta(\dot{x}^\alpha). \quad (\text{B3})$$

However as already noticed in [8], being spacetime flat we have  $\dot{p}_\mu = 0$ , and then the second term of last equation is just a total derivative:

$$p_\mu\delta\dot{x}^\mu = \frac{d}{d\tau}(p_\mu\delta x^\mu) = \frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\delta x^\mu\right), \quad (\text{B4})$$

which can be eliminated during the integration procedure. The Killing equation of our theory is then equivalent to (23), being

$$\delta(\tilde{g}_{\mu\nu}dx^\mu dx^\nu) = 0, \quad (\text{B5})$$

excepted total derivative terms.

Given this, from Killing equation (23) we find the system of differential equations we need to solve in order to find our symmetry generators:

$$\begin{cases} \tilde{g}_{0\mu}\partial_0\xi^\mu + \frac{1}{2}\frac{\partial\tilde{g}_{00}}{\partial\dot{x}^0}\partial_\alpha\xi^0\dot{x}^\alpha = 0 \\ \tilde{g}_{1\mu}\partial_1\xi^\mu + \frac{1}{2}\frac{\partial\tilde{g}_{11}}{\partial\dot{x}^0}\partial_\alpha\xi^0\dot{x}^\alpha = 0 \\ \tilde{g}_{\mu 1}\partial_0\xi^\mu + \tilde{g}_{0\nu}\partial_1\xi^\nu + \frac{\partial\tilde{g}_{01}}{\partial\dot{x}^1}\partial_\alpha\xi^1\dot{x}^\alpha = 0 \end{cases}. \quad (\text{B6})$$

We look for perturbative solutions at first order in  $\ell$  such as

$$\xi^0 = \xi_{(0)}^0 + \ell\xi_{(\ell)}^0, \quad \xi^1 = \xi_{(0)}^1 + \ell\xi_{(\ell)}^1, \quad (\text{B7})$$

in which the 0th order solutions are the Minkowskian ones:

$$\xi_{(0)}^0 = c^0 + kx^1, \quad \xi_{(0)}^1 = c^1 + kx^0. \quad (\text{B8})$$

This leads to

$$\begin{cases} \partial_0\xi_\ell^0 = \left(\frac{\gamma}{2} + \beta_2\right)k\dot{x}^1 \\ \partial_1\xi_\ell^1 = -\left(\beta_2 + \frac{\beta_1}{2}\right)k\dot{x}^1 \\ \partial_0\xi_\ell^1 - \partial_1\xi_\ell^0 = -(\beta_1 + \beta_2 + \gamma)k\dot{x}^0 \end{cases}. \quad (\text{B9})$$

From system (B9) first and second equation we just obtain that the generalized Killing vector components should have the form:

$$\xi_{(\ell)}^0 = \kappa^0(x^1, \dot{x}) + \left(\frac{\gamma}{2} + \beta_2\right)\dot{x}^1x^0, \quad \xi_{(\ell)}^1 = \kappa^1(x^0, \dot{x}) - \left(\beta_2 + \frac{\beta_1}{2}\right)\dot{x}^1x^1. \quad (\text{B10})$$

Now using the results obtained (B10) with the third equation, re-deriving this one we get

$$\kappa^0(x^1, \dot{x}) = \mathbb{C}^0(\dot{x}) + \kappa^0(\dot{x})x^1, \quad (\text{B11})$$

$$\kappa^1(x^0, \dot{x}) = \mathbb{C}^1(\dot{x}) + \kappa^1(\dot{x})x^0. \quad (\text{B12})$$

Then, using (B12) again with the third equation of system (B9) we obtain the relation between  $\kappa^0$  and  $\kappa^1$  which is:

$$\kappa^0(\dot{x}) = \kappa^1(\dot{x}) + (\beta_1 + \beta_2 + \gamma)k\dot{x}^0. \quad (\text{B13})$$

We can now express the complete solutions for the  $\xi_{(\ell)}^\alpha$  as

$$\xi_{(\ell)}^0 = \mathbb{C}^0(\dot{x}) + \kappa^1(\dot{x})x^1 + k\left((\beta_1 + \beta_2 + \gamma)\dot{x}^0x^1 + \left(\frac{\gamma}{2} + \beta_2\right)\dot{x}^1x^0\right), \quad (\text{B14})$$

$$\xi_{(\ell)}^1 = \mathbb{C}^1(\dot{x}) + \kappa^1(\dot{x})x^0 - k\left(\beta_2 + \frac{\beta_1}{2}\right)\dot{x}^1x^1. \quad (\text{B15})$$

The natural choice  $\mathbb{C}^0(\dot{x}) = \mathbb{C}^1(\dot{x}) = \kappa^1(\dot{x}) = 0$  gives us the total charge, in fact it is easy to verify that in this case:

$$\begin{aligned} Q_\ell = \xi^\mu p_\mu &= c^0 p_0 + c^1 p_1 + k\left(x^1 p_0 + (\beta_1 + \beta_2 + \gamma)\ell x^1 \dot{x}^0 p_0 - \frac{\ell}{2}(\beta_1 + 2\beta_2)x^1 \dot{x} p_1 + x^0 p_1 + \ell\left(\frac{\gamma}{2} + \beta_2\right)x^0 \dot{x}^1 p_0\right) \simeq \\ &\simeq c^0 p_0 + c^1 p_1 + k\left(x^0 p_1\left(1 - \ell\left(\beta_2 + \frac{\gamma}{2}\right)p_0\right) + x^1\left(p_0 + (\beta_1 + \beta_2 + \gamma)\ell p_0^2 + \frac{\ell}{2}(\beta_1 + 2\beta_2)p_1^2\right)\right). \end{aligned} \quad (\text{B16})$$

The choice  $\kappa^1(\dot{x}) = 0$  works fine in the diagonal case metric. However Finsler formalism relies on homogeneous metrics, which in general have non zero off-diagonal terms. The total charge general expression imposing no particular condition to our parameters at first order in  $\ell$  has the form

$$Q_\ell = \xi^\mu p_\mu \simeq (c^0 + \ell \mathbb{C}^0(\dot{x}))p_0 + (c^1 + \ell \mathbb{C}^1(\dot{x}))p_1 + (k + \ell \kappa^1(\dot{x}))\mathcal{N}, \quad (\text{B17})$$

in which  $p_0$ ,  $p_1$  and  $\mathcal{N}$  are the deformed-Poincaré algebra generators in 1+1D. In order to obtain a coherent physical framework (*id est* invariant Casimir and particles worldlines) also in the Finsler case, we need then to impose  $\kappa^1(\dot{x}) = (\beta_2 - \frac{\gamma}{2})\dot{x}^0$ . This choice eliminates any direct sign of the metrics off-diagonal terms in the boost representation (25).

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